Topic - Polar Equation of Conic Section

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Equation of the Tangent and Normal at the point α of the Conic

$$\frac{l}{r} = 1 + e\cos\theta$$

Find the Equation of the Tangent at the point α of the Conic Section

$$\frac{l}{r} = 1 + e\cos\theta$$

Solution → Since we know the Equation of the Chord joining the point $P(\alpha)$ and $Q(\beta)$ is

$$\frac{l}{r} = \sec\frac{\beta-\alpha}{2}\cos\left[\theta - \frac{\beta+\alpha}{2}\right] + e\cos\theta.$$

this chord ~~which is the~~ becomes Tangent at the point $P(\alpha)$, if the point $Q(\beta)$ Coincides with the point $P(\alpha)$ i.e if $\beta \to \alpha$

thus the Equation

$$\frac{l}{r} = \sec 0 \cos(\theta-\alpha) + e\cos\theta$$

$$= \cos(\theta-\alpha) + e\cos\theta.$$

Corollary → The Tangent at α to the Conic

$\frac{l}{r} = 1 - e\cos\theta$ is $l/r = \cos(\theta-\alpha) - e\cos\theta.$

Equation of Normal at the point $P(\alpha)$ to the Conic

$$\frac{l}{r} = 1 + e\cos\theta.$$

The Equation of the Conic is

$$\frac{l}{r} = 1 + e\cos\theta \quad \text{——— (1)}$$

The Equation of tangent at point α to the conic $l/r = 1 + e\cos\theta$ is Given by

$$\frac{l}{r} = \cos(\theta - \alpha) + e\cos\theta \quad \text{——— (2)}$$

Equation of any line $\perp$ to (2) at α is

$$\frac{Kl}{r} = \cos\left(\theta + \frac{\pi}{2} - \alpha\right) + e\cos(\theta + \pi/2) \longrightarrow (3)$$

On Changing θ in to $\theta + \pi/2$

(where K = constant

Since (3) passes through α

i.e $\left(\frac{l}{1 + e\cos\alpha}, \alpha\right)$ therefore

$$K(1 + e\cos\alpha) = \cos\pi/2 + e\cos(\alpha + \pi/2)$$
$$= 0 - e\sin\alpha$$

$$\therefore K = -\frac{e\sin\alpha}{1 + e\cos\alpha}$$

Hence the Normal at α to (1) is

$$= \frac{-e\sin\alpha}{r(1+e\cos\alpha)} = -\sin(\theta-\alpha) - e\sin\alpha$$

By 3

$$\text{or } \frac{l\, e\sin\alpha}{r(1+e\cos\alpha)} = \sin(\theta-\alpha) + e\sin\alpha$$

Equation of Asymptotes of the Conic

$$\frac{l}{r} = 1 + e\cos\theta$$

Here Equation of Conic is

$$\frac{l}{r} = 1 + e\cos\theta \quad \text{--- (1)}$$

The equation of tangent to the Conic (1) at the point α is

$$\frac{l}{r} = e\cos\theta + \cos(\theta-\alpha)$$

The Asymptote is a tangent to the Conic at infinity

i.e the point α is a point at infinity on the Conic, if

$$0 = 1 + e\cos\alpha \qquad \text{or } \cos\alpha = -\tfrac{1}{e}$$

$$\therefore \sin\alpha = \pm\sqrt{1 - \tfrac{1}{e^2}} =$$

Eliminating α from (2) we get

$$\frac{l}{r} = e\cos\theta + \cos\theta \cos\alpha + \sin\theta\sin\alpha$$

$$= e\cos\theta + \cos\theta\left(-\tfrac{1}{e}\right) \pm \sin\theta\sqrt{1-\tfrac{1}{e^2}}$$

$$= \left(e - \tfrac{1}{e}\right)\cos\theta \pm \sin\theta\frac{\sqrt{(e^2-1)}}{e}$$

$$\Rightarrow \frac{el}{r} = (e^2-1)\cos\theta \pm \sqrt{e^2-1}\,\sin\theta$$

This is the required equation of Asymptotes

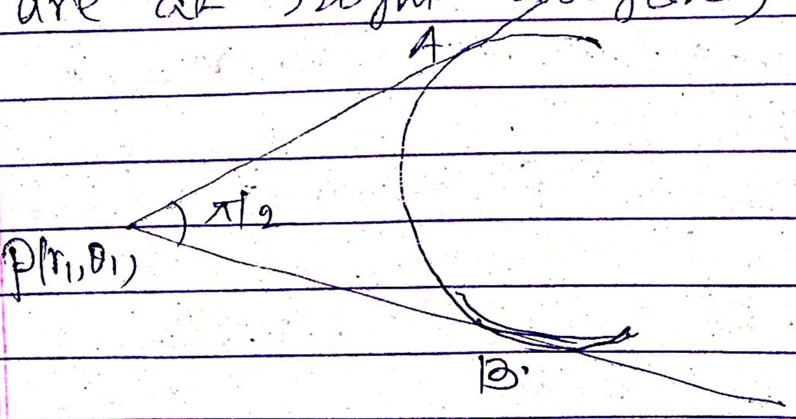
* Equation of the Director Circle of the Conic

$$\frac{l}{r} = 1 + e\cos\theta$$

Since Equation of the conic is

$$\frac{l}{r} = 1 + e\cos\theta$$

[The Director Circle is the Locus of points the tangent from which to the conic are at right angles)



Draw perpendicular tangents to the Conic at the points $A(\alpha)$ and $B(\beta)$

Then the Equations of the Tangents to the Conic $A(\alpha)$ and $B(\beta)$ are given by respectively

$$\frac{l}{r} = e\cos\theta + \cos(\theta - \alpha) \quad \text{--- (2)}$$

$$l/r = e\cos\theta + \cos(\theta - \beta) \quad \text{--- (3)}$$

If these two tangents intersect at $P(r_1, \theta_1)$ then

$$\frac{l}{r_1} = e\cos\theta_1 + \cos(\theta_1 - \alpha) \quad \text{--- (4)}$$

$$\frac{l}{r_1} = e\cos\theta_1 + \cos(\theta_1 - \beta) \quad \text{--- (5)}$$

To get r_1 and θ_1

(4) − (5)

$$\cos(\theta_1 - \alpha) = \cos(\theta_1 - \beta)$$

$$\theta_1 - \alpha = \pm(\theta_1 - \beta)$$

the Lower sign must be taken here since upper sign would give a special value of β.

$$\therefore \theta_1 - \alpha = -\theta_1 + \beta \quad \text{--- (6)}$$

$$\Rightarrow \theta_1 = \frac{\alpha + \beta}{2}$$

From (4)

$$\frac{l}{r} - e\cos\theta_1 = \cos\left(\frac{\alpha+\beta}{2} - \alpha\right)$$

$$= \cos\left(\frac{\beta - \alpha}{2}\right) \quad \text{--- (7)}$$

From (2)

$$l/r = e\cos\theta + \cos\theta\cos\alpha + \sin\theta\sin\alpha$$

$$l = r\cos\theta(e + \cos\alpha) + r\sin\theta\sin\alpha$$

But $r\cos\theta = x$ and $r\sin\theta = y$.

$$\therefore l = x(e + \cos\alpha) + y\sin\alpha.$$

The slope of the Tangent (2)

$$= -\frac{\text{The co-efficient of } x}{\text{The co-efficient of } y}$$

$$= -\frac{e+\cos\alpha}{\sin\alpha}$$

Similarly the slope of the Tangent 3

$$= -\frac{e+\cos\beta}{\sin\beta}$$

Since these two tangents are mutual perpendicular, therefore,

$$\left(-\frac{e+\cos\alpha}{\sin\alpha}\right)\left(-\frac{e+\cos\beta}{\sin\beta}\right) = -1$$

$$\Rightarrow (e+\cos\alpha)(e+\cos\beta) + \sin\alpha\sin\beta = 0$$

$$\Rightarrow e^2 + e\cos\beta + e\cos\alpha + \cos\alpha\cos\beta + \sin\alpha\sin\beta = 0$$

$$\Rightarrow e^2 + e(\cos\alpha + \cos\beta) + \cos(\alpha-\beta) = 0$$

$$\Rightarrow e^2 + e \times 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} + 2\cos^2\frac{\alpha-\beta}{2} - 1 = 0$$

$$\Rightarrow e^2 + 2e\cos\theta_1\left(\frac{l}{r_1} - e\cos\theta_1\right) + 2\cos^2\frac{\alpha-\beta}{2} - 1 = 0$$

$$\Rightarrow e^2 + 2e\cos\theta_1\left(\frac{l}{r_1} - e\cos\theta_1\right) + 2\left(\frac{l}{r_1} - e\cos\theta_1\right)^2 - 1 = 0$$

$$\Rightarrow r_1^2e^2 + 2le\,r_1\cos\theta - 2r_1^2e^2\cos^2\theta_1 + 2l^2 - 4el\,r_1\cos\theta_1$$
$$+ 2r_1^2e^2\cos^2\theta_1 - r_1^2 = 0$$

$$\Rightarrow (1-e^2)r_1^2 + 2el\,r_1\cos\theta_1 - 2l^2 = 0$$

Thus the locus of (r_1, θ_1) is
$(1-e^2)r^2 + 2elr\cos\theta - 2l^2 = 0$ This is the equation of the Director circle.